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Relativistic theory of magnetoelastic interactions I

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Abstract. In the first part of the present work, the relativistically invariant field equations relevant to mechanics, thermodynamics and electromagnetism are presented in a phenomenological continuum theory of magnetoelastic interactions in which the magnetic spin is accounted for.

1. Introduction

In a recent series of papers, Maugin and Eringen (1972a, 1972b, 1972c, also Maugin 1971a) have formulated a continuum theory of magnetoelastic interactions in which the spin of magnetic origin is accounted for. This kind of approach is often said to resort to *micromagnetism* or *micromagnetic* theory. The theory was intended to be applied to ferromagnetic materials and includes, in a phenomenological way, effects that can be expected in these materials, such as the existence of an anisotropy field and of the exchange energy, the latter due to the interactions developed between neighbouring spins. However, the theory developed was still incomplete. In Maugin and Eringen (1972a, 1972b) the treatment was three dimensional with the electric part of the electromagnetic field completely ignored. Moreover we neglected the currents and, though we took into consideration dynamical terms as far as mechanics was concerned, the magnetic part was treated in quasistatics. Dissipation processes and loss mechanisms have been discarded for most part of the works quoted above since we used a variational formulation to start with. The theory so given presented the same incompleteness as that of Brown (1966) and Tiersten (1965). A special kind of dissipation derivable from a Rayleigh potential has been considered in Maugin and Eringen (1972a). It is similar to that introduced earlier by Gilbert and Kelley (1955) and was shown to be equivalent to that of Landau and Lifshitz (1935). It has also been considered by Kaliski (1969) who closely follows the general exposé given by Akhiezer *et al* (1967). A larger class of dissipative processes has been dealt with by Maugin and Eringen (1972a, § 8), Tiersten (1964) and Kaliski (1969). Maugin and Eringen make use of the thermodynamical admissibility (Clausius–Duhem inequality). In contrast to the variational formulation referred to above, the field equations were there postulated in global form (ie for the whole body under study) along with the second principle of thermodynamics.

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It was further shown by Maugin and Eringen (1972c) that a fully dynamical theory could be formulated without too much sophistication with the universe of Minkowski as background. The latter work was in fact an extension of the works of Grot and Eringen (1966) and Grot (1970) in special relativity and of Maugin (1971f) in general relativity, with a view to including the magnetic spin and the magnetization gradients in the framework of the theory. A variational principle was still used and, consequently, no dissipative processes were considered. The current was solely due to convection; the Joule term $\boldsymbol{J}.\boldsymbol{E}$ thus vanished in a rest frame. We must note here that the theory thus constructed provided a generalization of the earlier results of Frenkel (1926), Mathison (1937) and Weyssenhoff and Raabe (1947) (concerning the theory of the spinning electron) to deformable media by introducing a general stress tensor and the magnetization gradients. It also constitutes the counterpart, for nonlinear magnetized elastic solids, of the theory of spinning fluids developed by Halbwachs (1960) in his monograph. It should be pointed out that the works of Frenkel, Mathison, Weyssenhoff, Raabe and Halbwachs are concerned with the quantum theory of Dirac's electron. They have been used for the last two decades by physicists who favour a causal reinterpretation of quantum mechanics and consider nonlinear field equations based on the *hydrodynamical* interpretation of the wavefunction (see particularly, Halbwachs 1960 and references quoted therein). It is therefore no wonder that the equations developed in relativistic continuum mechanics present great similarities with those of this quantum theory. Note that, in contrast to the recent article of Ellis (1971), the present work and the preceding ones of Maugin and Eringen deal with a *continuum* description; the infinitesimal element of matter playing here the role of the point particle considered by Ellis. The number of degrees of freedom is thus largely increased, in fact it is infinite for a continuous medium. We therefore take account of strains, finite for the sake of generality, and consequently of stresses. A linear theory may, of course, be deduced for suitable approximations.

After this short critical review, we can state that a fully dynamical theory of magnetoelastic interactions including the effects of spin, magnetization and polarization gradients, currents and an arbitrary form of dissipation should possibly be obtained in the frame of special relativity by *postulating* global balance laws subject to thermodynamical restrictions. This is tentatively done in the subsequent sections. Yet, to simplify the exposé, we shall disregard the *orientation* of the material due to purely structural causes. That is, the theory given is *not* constructed for *polar* media (for relativistic polar continua, see Kafadar and Eringen (1971) and Maugin and Eringen 1972d). The orientation of the element of matter (assimilated to a small magnet) results here only from the magnetic spin. That the two effects, magnetic orientation and structural orientation, are comparable is well known from the theory of liquid crystals. The structure of the field equations for the two cases has been looked at by Maugin and Eringen (1972a). How these two effects can be brought together has been examined by Maugin (1971e). We shall therefore not come back to this point.

A last point must be emphasized before we turn to the detailed analysis. That we must consider a large class of dissipative processes in this phenomenological theory results from the very existence of complicated loss mechanisms in the theory of ferromagnetic materials. For instance, from a number of applications in ferromagnetic resonance, it is known that the magnetization vector spirals into parallelism with the applied magnetic field. This fact is given a good representation by introducing a damping term in the spin equation. Furthermore there exist numerous experimental investigations of the dissipation of energy in domain wall propagation (whose study resorts to micromagnetism) and in coherent rotation phenomena. The nature of these dissipations is still unclear. This is certainly a sufficient reason to study linear and nonlinear dissipation processes from a phenomenological point of view.

The present exposé is in the realm of classical continuum physics. Of course, it remains to treat a number of practical problems and to compare the results with experimental data in order to assert the physical validity of this phenomenological theory. There exist other possible approaches to the problem of micromagnetism, either of the *micromorphic* type (cf Eringen and Kafadar 1970 and Eringen 1971b) or of the *geometrical* type (cf Maugin 1971d). They should be promising but, still in infancy, they need to be unequivocally linked to the physics of magnetic phenomena.

Notations and basic kinematical and geometrical notions are recalled in § 2. In § 3, we state the *global* form of balance laws for hypersurfaces embedded in the minkowskian space–time manifold in a somewhat axiomatic manner. This is done for mechanics, electromagnetism and thermodynamics. When a proper physical significance is given to each of the terms which appear in these equations, we deduce the local form of the field equations. In § 4, the field equations although written in four dimensional formalism, are given in a form very similar to that of three dimensional physics. In § 5, we examine the constraints imposed upon the field equations when the magnetization is saturated. We thus obtain special possible forms of the equation governing the magnetic spin. The system of equations given in this part of the work is underdetermined. Constitutive equations for a particular material required to close this system will be developed in part II after a thorough study of local forms of the second principle of thermodynamics (Clausius–Duhem inequality) appropriate to the consideration of dissipative processes.

2. Preliminaries

The notation used hereafter is closely related to that of Grot and Eringen (1966), Maugin and Eringen (1972c, 1972d) and Maugin (1971b, 1971c, 1971d, 1971f). We use the standard tensor notation and summation convention of diagonally repeated indices. Greek indices (small or capital) assume the values 1, 2, 3 and 4 while all Latin indices take the values 1, 2, 3. Parentheses around a set of indices denote symmetrization and brackets denote alternation. $\epsilon^{\alpha\beta\gamma\delta}$ is the permutation symbol whose algebra is based on the formulae

$$\epsilon^{\mu\nu\sigma\rho}\epsilon_{\mu\nu\sigma\tau} = 3!\delta^{\rho}_{\tau}$$
$$\epsilon^{\mu\nu\sigma\tau}\epsilon_{\mu\nu\alpha\beta} = 2\delta^{\sigma\tau}_{\alpha\beta} = 4\delta^{\sigma}_{[\alpha}\delta^{\tau}_{\beta]}$$

where δ^{σ}_{α} is the Kronecker symbol. Only a very brief account of kinematics is given here. The reader is referred to the authors quoted above for a more extensive presentation.

In the Minkowski four dimensional space–time manifold M^4, the squared arc length element is written in a so called *inertial* frame of reference as

$$(\mathrm{d}s)^2 = (\mathrm{d}x)^2+(\mathrm{d}y)^2+(\mathrm{d}z)^2-c^2(\mathrm{d}t)^2 = \mathrm{d}z_\alpha\,\mathrm{d}z^\alpha$$
$$(z^1, z^2, z^3, z^4) = (x, y, z, \mathrm{i}ct) \qquad \mathrm{i} = (-1)^{1/2} \tag{2.1}$$

where x, y, z are rectangular coordinates in euclidean space E^3, t is the newtonian absolute time and c is the velocity of light in vacuum. In a *curvilinear* frame of coordinates x^α (x^4 timelike), the metric $g_{\alpha\beta}$ is symmetric and normal hyperbolic, that is, of signature $(+, +, +, -)$ and equation (2.1) reads

$$(\mathrm{d}s)^2 = g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta$$

with

$$g_{\alpha\beta}g^{\beta\gamma} = \delta^\gamma_\alpha \qquad g = \det(g_{\alpha\beta}) \tag{2.2}$$

where $g^{\beta\gamma}$ is the reciprocal of $g_{\alpha\beta}$.

The *direct* motion of a material particle labelled (X^K) along its world line $(\mathscr{C}_{X^K})$ is entirely described by the mapping of class C^2

$$x^\alpha = \mathscr{X}^\alpha(X^\Delta) \qquad X^\Delta = (X^K, \mathrm{i}c\tau) \tag{2.3}$$

where X^K are lagrangian coordinates in euclidean reference space E^3_R and τ is the proper time of (X^K). The operator $\partial/\partial\tau$ (also denoted by a superscript dot) is the differentiation with respect to τ; hence it defines rates. The four velocity u^α and the four acceleration $\dot{u}^\alpha$ are defined by

$$u^\alpha = \dot{x}^\alpha = \frac{\partial x^\alpha}{\partial\tau} \tag{2.4}$$

with

$$u^\alpha u_\alpha = -c^2 \qquad u_\alpha \nabla_\beta u^\alpha = 0$$

$$\dot{u}^\alpha = \ddot{x}^\alpha = \frac{\partial^2 x^\alpha}{\partial\tau^2} \tag{2.5}$$

with

$$\dot{u}^\alpha u_\alpha = 0.$$

A comma or a symbol ∂ followed by an index indicates partial derivative. A semicolon or a symbol ∇ followed by an index denotes covariant partial derivative based on the $g_{\alpha\beta}$.

X^K and τ are independent variables such that we can invert equation (2.3) to get

$$X^K = X^K(x^\alpha) \qquad \tau = \tau(x^\alpha)$$

with (2.6)

$$\frac{\partial\tau}{\partial X^K} = \frac{\partial X_K}{\partial\tau} = 0.$$

It follows that:

$$x^\alpha{}_{,K} \equiv \frac{\partial x^\alpha}{\partial X^K} \qquad X^K{}_{,\alpha} = \frac{\partial X^K}{\partial x^\alpha} \tag{2.7}$$

are well defined quantities. The chain rule of differentiation yields

$$X^K{}_{,\mu}x^\mu{}_{,L} = \delta^K_L \qquad x^\mu{}_{,K}X^K{}_{,\lambda} = \delta^\mu_\lambda - u^\mu\tau_{,\lambda} \tag{2.8}$$

where δ^K_L is the Kronecker symbol in E^3_R.

Of great importance in the subsequent developments, is the *projector* or projection operator $P_{\alpha\beta}$. Defined at an event point $\boldsymbol{M}$ in M^4 as

$$P_{\alpha\beta}(\boldsymbol{M}) = g_{\alpha\beta} + \frac{1}{c^2}u_\alpha u_\beta \qquad P^\alpha_{.\alpha} = 3 \tag{2.9}$$

it allows the decomposition of any tensor along the four velocity and onto the hypersurface $M_\perp^3$ orthogonal to $(\mathscr{C}_{XK})$ at $\boldsymbol{M}$. Given a four vector field f^α at $\boldsymbol{M}$, we have

$$f^\alpha = \overline{f^\alpha} + fu^\alpha \tag{2.10}$$

with

$$\left.\begin{array}{ll} \overline{f^\alpha} \equiv P^\alpha_{\cdot\beta} f^\beta & \overline{f^\alpha} \subset M_\perp^3 \\ f \equiv -\dfrac{1}{c^2} f^\alpha u_\alpha & fu^\alpha // u^\alpha. \end{array}\right\} \tag{2.11}$$

If in equation (2.10), $f \equiv 0$, then $f^\alpha \equiv \overline{f^\alpha}$. We say that f^α is a PU four vector field (ie perpendicular to u^α). f^α is in $M_\perp^3$. More generally, a tensor of any order $A^{\alpha\beta\ldots\mu}$ is said to be PU if and only if

$$A^{\alpha\beta\ldots\mu} u_\alpha = A^{\alpha\beta\ldots\mu} u_\beta = A^{\alpha\beta\ldots\mu} u_\mu = 0 \tag{2.12}$$

then

$$\overline{A^{\alpha\beta\ldots\mu}} = P^\alpha_{\cdot\sigma} P^\beta_{\cdot\rho} \ldots P^\mu_{\cdot\nu} A^{\sigma\rho\ldots\nu} \equiv A^{\alpha\beta\ldots\mu}.$$

For a second order symmetric or skewsymmetric tensor $A^{\alpha\beta}$, we have

$$A^{\alpha\beta} u_\beta = 0 \rightarrow A^{\alpha\beta} \text{ is PU} \qquad \overline{A^{\alpha\beta}} \equiv A^{\alpha\beta}. \tag{2.13}$$

We remark that $P_{\alpha\beta}$ is symmetric, idempotent and PU. That is

$$P_{\alpha\beta} = P_{\beta\alpha} \qquad P^\alpha_{\cdot\beta} P^\beta_{\cdot\gamma} = P^\alpha_{\cdot\gamma} \qquad P_{\alpha\beta} u^\beta = 0.$$

A general second order tensor $T^{\mu\nu}$ is decomposed according to the formula

$$\left.\begin{array}{ll} T^{\mu\nu} = au^\mu u^\nu + a_1^\mu u^\nu + a_2^\nu u^\mu + a^{\mu\nu} & \\ a_1^\mu u_\mu = a_2^\mu u_\mu = 0 & a^{\mu\nu} u_\mu = a^{\nu\mu} u_\mu = 0. \end{array}\right\} \tag{2.14}$$

Note that a second order skewsymmetric tensor $A^{\alpha\beta}$ has a decomposition of the form

$$A_{\alpha\beta} = \frac{2}{c} A_{[\alpha} u_{\beta]} + \overset{*}{A}_{\alpha\beta} \tag{2.15}$$

where

$$\left.\begin{array}{lll} A_\beta \equiv -\dfrac{1}{c} A_{\alpha\beta} u^\beta & A_\beta u^\beta = 0 & \\ \overset{*}{A}_{\alpha\beta} = P^{\cdot\mu}_{\alpha} P^{\cdot\nu}_{\beta} A_{\mu\nu} & \overset{*}{A}_{\alpha\beta} = \dfrac{1}{\mathrm{i}c} \epsilon_{\alpha\beta\gamma\delta} a^\gamma u^\delta & \\ a^\alpha \equiv \dfrac{1}{2\mathrm{i}c} \epsilon^{\alpha\beta\gamma\delta} \overset{*}{A}_{\beta\gamma} u_\delta & \overset{*}{A}_{\alpha\beta} u^\beta = 0 & a^\alpha u_\alpha = 0 \end{array}\right\} \tag{2.16}$$

where A_β and a^α are PU.

Before defining the strain measures, we recall the following postulates (Maugin 1971c):

(i) All tensorial objects used for describing the deformation field of a continuous medium in M^4 must be either defined in E_R^3 (ie are lagrangian tensors) or PU in M^4.

From which it follows:

(ii) All tensorial objects used for describing the deformation field of a continuous medium in M^4 must reduce to their analogues of classical continuum mechanics in a rest frame.

Postulate (ii) is a straightforward consequence of postulate (i). These postulates serve as guidelines for the choice of correct measures of deformation in M^4. We now recall the definition of the strain measures and of their rates needed in the following. The direct and inverse deformation gradients and the relativistic Green tensor C_{KL} and its reciprocal $\overset{-1}{C}{}^{KL}$ are defined by

$$x^{\alpha}_{.K} = \overline{x^{\alpha}_{,K}} \qquad X^{K}_{,\alpha} = \frac{\partial X^K}{\partial x^{\alpha}} \tag{2.17}$$

$$C_{KL} \equiv P_{\alpha\beta} x^{\alpha}_{.K} x^{\beta}_{.L} \qquad \overset{-1}{C}{}^{KL} C_{LM} = \delta^K_M . \tag{2.18}$$

Then

$$x^{\alpha}_{.K} X^{K}_{,\beta} = P^{\alpha}_{.\beta} \qquad X^{K}_{,\mu} x^{\mu}_{.L} = \delta^K_L . \tag{2.19}$$

In the following, $\sigma_{\alpha\beta}$ and $\omega_{\alpha\beta}$ are the rate of strain tensor and the vorticity tensor respectively, with

$$\left.\begin{aligned} &\sigma_{\alpha\beta} = e_{(\alpha\beta)} \qquad \omega_{\alpha\beta} = e_{[\alpha\beta]} \\ &e_{\alpha\beta} \equiv P^{\cdot\mu}_{\alpha} \nabla_{\mu} u_{\gamma} P^{\gamma}_{.\beta} . \end{aligned}\right\} \tag{2.20}$$

We can thus complete the following diagram of useful strain measures and rates of deformation:

$$\begin{tikzcd}[column sep=huge]
 & \text{in } E^3_{\mathrm{R}} \arrow[d] & \text{in } M^3_{\perp} \arrow[d] \\
\text{Strain measures} & C_{KL} \arrow[r, shift right, "P_{\alpha\beta} = C_{KL} X^K_{,\alpha} X^L_{,\beta}"'] \arrow[d, "\frac{\partial}{\partial \tau}"'] & P_{\alpha\beta} \arrow[l, shift right, "C_{KL} = P_{\alpha\beta} x^{\alpha}_{.K} x^{\beta}_{.L}"'] \arrow[d, "\frac{1}{2}\pounds_u"] \\
\text{Rates} & \dot{C}_{KL} \arrow[r, shift right, "\sigma_{\alpha\beta} = \frac{1}{2} \dot{C}_{KL} X^K_{,\alpha} X^L_{,\beta}"'] & \sigma_{\alpha\beta} \arrow[l, shift right, "\dot{C}_{KL} = 2\sigma_{\alpha\beta} x^{\alpha}_{.K} x^{\beta}_{.L}"']
\end{tikzcd}$$

where $\pounds_u$ indicates the Lie derivative with respect to the four vector field u^{α} (cf Maugin 1971b). It is straightforward to verify that $x^{\alpha}_{.K}$, $X^{K}_{,\alpha}$, $e_{\alpha\beta}$, $\sigma_{\alpha\beta}$ and $\omega_{\alpha\beta}$ are PU tensor fields, hence 'good' measures (cf postulate (i)).

The last notations we shall need are those related to hypersurfaces embedded in M^4 and to integrals over these. We denote by $\mathrm{d}v_4$, $\mathrm{d}s_{3\mu}$, $\mathrm{d}s_2^{\alpha\beta}$ and $\mathrm{d}\hat{s}_{2\alpha\beta}$ the four element of volume in M^4, the oriented three dimensional surface element of a three dimensional hypersurface $(\mathscr{S}^3)$ (whose oriented unit normal is n_{μ}) and the two dimensional surface element and its dual, of a two dimensional hypersurface $(\mathscr{S}^2)$ (cf Grot and Eringen 1966). We have

$$\begin{aligned} dv_4 &= \sqrt{|g|}\, dx^1\, dx^2\, dx^3\, dx^4 && \text{in the } x^\alpha \text{ system} \\ &= ic\, dX^1\, dX^2\, dX^3\, d\tau && \text{in the } X^\Delta \text{ system} \\ ds_{3\mu} &= n_\mu\, ds_3 . \end{aligned} \tag{2.21}$$

Gaussian parametrizations are assumed on $(\mathscr{S}^3)$ and $(\mathscr{S}^2)$ in order to define surface elements. Finally, let $(\partial\mathscr{B})$ be a three dimensional hypersurface, regular† boundary of a region $(\mathscr{B})$ of M^4. (Γ) is a regular† discontinuity three dimensional hypersurface in $(\mathscr{B})$. Let ϕ^α be a four vector field. With the obvious notation $\Phi[\partial\mathscr{B}]$, we have the generalized Green–Gauss theorem (cf Eringen 1971a, p 125):

$$\Phi[\partial\mathscr{B}] = \int_{(\partial\mathscr{B}-\Gamma)} \phi^\alpha\, ds_{3\alpha} = \int_{(\mathscr{B}-\Gamma)} \phi^\alpha{}_{;\alpha}\, dv_4 + \int_{(\Gamma)} [\phi^\alpha]\, ds_{3\alpha}(\Gamma) \tag{2.22}$$

where the familiar symbolism $[\ldots]$ denotes the jump across (Γ).

3. Balance laws

3.1. Global balance laws

We postulate the global balance laws for the theory of magnetoelastic interactions in the following manner. To every three dimensional hypersurface $(\mathscr{S}^3)$ in M^4, we assign a positive scalar invariant R, a four vector P^α, a skewsymmetric tensor $\mathfrak{M}^{\alpha\beta}$, a scalar invariant H and a scalar function J. They are called the total mass flux, the total rate of energy–momentum, the total rate of moment of energy–momentum, the total rate of entropy and the total flux of charge respectively. To every two dimensional subspace $(\mathscr{S}^2)$ in M^4, we assign two scalar invariants F and Γ. All these tensorial quantities are postulated to transform as Lorentz fields. They are written in explicit form as

$$R[\mathscr{S}^3] = \int_{(\mathscr{S}^3)} \rho^\alpha\, ds_{3\alpha} \tag{3.1}$$

$$P^\alpha[\mathscr{S}^3] = \int_{(\mathscr{S}^3)} T^{\alpha\beta}\, ds_{3\beta} \tag{3.2}$$

$$\mathfrak{M}^{\alpha\beta}[\mathscr{S}^3] = \int_{(\mathscr{S}^3)} \mathbf{M}^{\alpha\beta\gamma}\, ds_{3\gamma} \tag{3.3}$$

with

$$\mathfrak{M}^{\alpha\beta} = -\mathfrak{M}^{\beta\alpha} \qquad \mathbf{M}^{(\alpha\beta)\gamma} = 0$$

$$H[\mathscr{S}^3] = \int_{(\mathscr{S}^3)} \eta^\alpha\, ds_{3\alpha} \tag{3.4}$$

$$J[\mathscr{S}^3] = \int_{(\mathscr{S}^3)} J^\alpha\, ds_{3\alpha} \tag{3.5}$$

$$F[\mathscr{S}^2] = \tfrac{1}{2}\int_{(\mathscr{S}^2)} F_{\alpha\beta}\, ds_2^{\alpha\beta} \qquad F_{\alpha\beta} = -F_{\beta\alpha} \tag{3.6}$$

† Smooth enough to avoid nonunique definitions of the oriented normal at each point.

$$\Gamma[\mathscr{S}^2] = \int_{(\mathscr{S}^2)} G^{\alpha\beta}\,\mathrm{d}\hat{s}_{2\alpha\beta} \qquad G^{\alpha\beta} = -G^{\beta\alpha}. \tag{3.7}$$

The tensor fields ρ^α, $T^{\alpha\beta}$, $\mathbf{M}^{\alpha\beta\gamma}$, η^α, J^α, $F_{\alpha\beta}$ and $G^{\alpha\beta}$ are called the mass flux four vector, the stress–energy–momentum tensor, the total spin tensor or angular momentum tensor, the total entropy flux four vector, the current four vector, the magnetic flux tensor and the electric displacement–magnetic field intensity tensor respectively.

Now define $\mathscr{F}^\alpha$ the four force per unit volume in M^4, $C^{\alpha\beta}$ the skewsymmetric tensor representing the couple per unit volume in M^4 and r the source of entropy per unit volume in M^4.

Let $(\mathscr{B})$ be a subregion of M^4 swept out by a material body (B) ((B_{R}) in the reference configuration in E^3_{R}) as time goes on, in the time interval $T = [t_1, t_2] \subset \mathbb{R}$, and $(\partial\mathscr{B})$ its boundary. We take $(\mathscr{S}^3) \equiv (\partial\mathscr{B})$. The global balance laws of mechanics, thermodynamics and electromagnetism are then stated as follows (cf Grot and Eringen 1966):

$$R[\partial\mathscr{B}] = 0 \tag{3.8}$$

$$P^\alpha[\partial\mathscr{B}] = \int_{(\mathscr{B})} \mathscr{F}^\alpha\,\mathrm{d}v_4 \tag{3.9}$$

$$\mathfrak{M}^{\alpha\beta}[\partial\mathscr{B}] = \int_{(\mathscr{B})} C^{\alpha\beta}\,\mathrm{d}v_4 \tag{3.10}$$

$$H[\partial\mathscr{B}] + \int_{(\mathscr{B})} r\,\mathrm{d}v_4 \geqslant 0 \tag{3.11}$$

$$J[\partial\mathscr{B}] = 0 \tag{3.12}$$

$$F[s^2] = 0 \tag{3.13}$$

$$\Gamma[\mathscr{S}^2] = J[\mathscr{S}^3]. \tag{3.14}$$

Equations (3.8) through (3.14) represent respectively the conservation of mass, the balance of energy–momentum, the balance of moment of energy–momentum, the second law of thermodynamics, the conservation of electric charge, the conservation of magnetic flux and the Ampère–Gauss laws. The two last equations are valid for every two dimensional circuit (s^2) in $(\mathscr{B})$ and outside $(\mathscr{B})$ and for every two dimensional circuit $(\mathscr{S}^2)$ enclosing a three dimensional subspace $(\mathscr{S}^3)$ within $(\mathscr{B})$ respectively.

These equations are supposed to be valid for any element of hypersurfaces for which they are written down. Thus, after application of Stokes' theorem and of the theorem (2.22) when necessary, with $\Sigma(x^\alpha) = 0$ a three dimensional discontinuity hypersurface present in $(\mathscr{B})$, we obtain the local field equations.

3.2. *Local balance laws*

They are

$$\rho^\alpha{}_{;\alpha} = 0 \text{ in } (\mathscr{B}-\Sigma) \qquad [\rho^\alpha]n_\alpha = 0 \text{ on } (\Sigma) \tag{3.15}$$

$$T^{\alpha\beta}{}_{;\beta} = \mathscr{F}^\alpha \text{ in } (\mathscr{B}-\Sigma) \qquad [T^{\alpha\beta}]n_\beta = 0 \text{ on } (\Sigma) \tag{3.16}$$

$$\mathbf{M}^{\alpha\beta\gamma}{}_{;\gamma} = C^{\alpha\beta} \text{ in } (\mathscr{B}-\Sigma) \qquad [\mathbf{M}^{\alpha\beta\gamma}]n_\gamma = 0 \text{ on } (\Sigma) \tag{3.17}$$

$$\eta^\alpha{}_{;\alpha} + r \geqslant 0 \text{ in } (\mathscr{B}-\Sigma) \qquad [\eta^\alpha]n_\alpha \geqslant 0 \text{ on } (\Sigma) \tag{3.18}$$

$$J^{\alpha}{}_{;\alpha} = 0 \text{ in } (\mathscr{B}-\Sigma) \qquad [J^{\alpha}]n_{\alpha} = 0 \text{ on } (\Sigma) \tag{3.19}$$

$$\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta;\beta} = 0 \text{ in } (\mathscr{B}-\Sigma) \qquad [\epsilon^{\alpha\beta\gamma\delta}F_{\beta\gamma}]n_{\delta} = 0 \text{ on } (\Sigma) \tag{3.20}$$

$$G^{\alpha\beta}{}_{;\beta} = J^{\alpha} \text{ in } (\mathscr{B}-\Sigma) \qquad [G^{\alpha\beta}]n_{\beta} = 0 \text{ on } (\Sigma) \tag{3.21}$$

where we have defined the unit normal to $\Sigma(x^{\alpha})$ by

$$n_{\alpha} \equiv \frac{\Sigma_{,\alpha}}{(|g^{\mu\nu}\Sigma_{,\mu}\Sigma_{,\nu}|)^{1/2}}. \tag{3.22}$$

For the sake of simplicity, we assumed that neither surface charge nor surface current were given on (Σ).

3.3. *Physical contents of local balance laws*

All tensorial quantities present in equations (3.15) through (3.21) admit unique decompositions according to the expressions (2.10), (2.14) and (2.15). Note that $T^{\alpha\beta}$ is a general second order tensor, $\mathbf{M}^{\alpha\beta\gamma}$ is skewsymmetric in α and β and $F_{\gamma\delta}$ and $G^{\alpha\beta}$ are skew-symmetric. The physical contents of the present theory are given by the physical significance granted to the elements of decomposition of these tensors. This is usually done by identifying term by term, in a cartesian frame of reference and at the limit $c \to \infty$, the space–space, mixed and time components of the above mentioned tensors with the physical objects which appear in the equations of classical three dimensional physics. This was done by Grot and Eringen (1966) and Kafadar and Eringen (1971). We shall not repeat these arguments which, in some cases, may be misleading. We only consider *a priori* special decompositions of the tensors in question—these decompositions have the only merit of yielding classical results in the proper 'nonrelativistic' limit (see eg § 11 of Maugin and Eringen 1972c). We shall take

$$\rho^{\alpha} = \rho u^{\alpha} \tag{3.23}$$

$$\left.\begin{aligned} &T^{\alpha\beta} = \omega u^{\alpha}u^{\beta} + \frac{1}{c^2}u^{\alpha}q^{\beta} + p^{\alpha}u^{\beta} - t^{\beta\alpha} \\ &q^{\alpha}u_{\alpha} = 0 \qquad p^{\alpha}u_{\alpha} = 0 \qquad t^{\beta\alpha}u_{\alpha} = u_{\beta}t^{\beta\alpha} = 0 \end{aligned}\right\} \tag{3.24}$$

$$\mathbf{M}^{\alpha\beta\gamma} = 2x^{[\alpha}T^{\beta]\gamma} + \mathscr{S}^{\alpha\beta\gamma} \tag{3.25}$$

$$\left.\begin{aligned} &\mathscr{S}^{\alpha\beta\gamma} = \rho S^{\alpha\beta}u^{\gamma} - 2M^{\alpha\beta\gamma} \\ &S^{\alpha\beta} = -S^{\beta\alpha} \quad S^{\alpha\beta}u_{\beta} = 0 \quad M^{\alpha\beta\gamma} = -M^{\beta\alpha\gamma} \quad M^{\alpha\beta\gamma}u_{\beta} = M^{\alpha\beta\gamma}u_{\gamma} = 0 \end{aligned}\right\} \tag{3.26}$$

$$\eta^{\alpha} = \overline{\eta^{\alpha}} + \rho\eta u^{\alpha} \qquad \text{with} \qquad \overline{\eta^{\alpha}} \equiv \frac{q^{\alpha}}{\theta} \tag{3.27}$$

$$J^{\alpha} = j^{\alpha} + \frac{q}{c^2}u^{\alpha} \qquad j^{\alpha} = \overline{J^{\alpha}} \qquad q = -J^{\alpha}u_{\alpha} \qquad j^{\alpha}u_{\alpha} = 0. \tag{3.28}$$

Here ρ is the relativistic invariant density of matter, a proper scalar field, that is, measured by an observer following the infinitesimal element of matter of mass ρ in its motion along $(\mathscr{C}_{XK})$. ω is the energy density per unit volume, q^{α} is the heat flux four vector, p^{α} is the nonmechanical momentum (by opposition with ωu^{α}) and $t^{\beta\alpha}$ is the relativistic stress tensor that represents macroscopically the matter–matter short range interactions which give rise to elastic forces. $S^{\alpha\beta}$ is the intrinsic spin per unit of proper mass, $M^{\alpha\beta\gamma}$ is the relativistic couple stress tensor, η is the proper scalar density of entropy and θ is the proper thermodynamical temperature. j^α is the conduction current while qu^α/c^2 is the convection current (q: volume charge density). The tensor fields q^α, p^α, $t^{\beta\alpha}$, $S^{\alpha\beta}$ and $M^{\alpha\beta\gamma}$ are obviously PU. ω is assumed to be the sum of the rest energy and of the so called relativistic internal energy $\mathscr{E}$. The latter, $\mathscr{E}$, must take account of the strain energy and, phenomenologically, of the interactions between matter and electromagnetic fields in $(\mathscr{B})$. For the proposed theory, it reflects the anisotropy and exchange energy. We write

$$\omega = \rho\left(1+\frac{\epsilon}{c^2}\right) \qquad \mathscr{E} = \rho\epsilon. \tag{3.29}$$

With the general decomposition (3.24), u^α is not in general an eigenvector of $T^{\alpha\beta}$ except if p^α and/or q^α are null.

According to equations (2.15)–(2.16), the fields $F_{\alpha\beta}$ and $G^{\alpha\beta}$ are written (cf Grot and Eringen 1966 and Lichnerowicz 1967)

$$\left.\begin{aligned} F_{\alpha\beta} &= \frac{1}{c}(\mathscr{E}_\beta u_\alpha - \mathscr{E}_\alpha u_\beta) + \frac{1}{\mathrm{i}c}\epsilon_{\alpha\beta\gamma\delta}\mathscr{B}^\gamma u^\delta \\ \mathscr{E}^\alpha &\equiv \frac{1}{c}F^{\alpha\beta}u_\beta \qquad \mathscr{E}^\alpha u_\alpha = 0 \qquad \mathscr{B}^\gamma u_\gamma = 0 \end{aligned}\right\} \tag{3.30}$$

$$\left.\begin{aligned} G^{\alpha\beta} &= \frac{1}{c}(\mathscr{D}^\beta u^\alpha - \mathscr{D}^\alpha u^\beta) + \frac{1}{\mathrm{i}c}\epsilon^{\alpha\beta\gamma\delta}\mathscr{H}_\gamma u_\delta \\ \mathscr{D}^\beta &\equiv \frac{1}{c}G^{\beta\alpha}u_\alpha \qquad \mathscr{D}^\beta u_\beta = 0 \qquad \mathscr{H}_\gamma u^\gamma = 0. \end{aligned}\right\} \tag{3.31}$$

The PU vector fields $\mathscr{E}^\alpha$, $\mathscr{B}^\gamma$, $\mathscr{D}^\beta$ and $\mathscr{H}_\gamma$ are the electric field four vector, the magnetic intensity four vector, the electric displacement four vector and the magnetic field four vector respectively. The relations existing between these four dimensional electromagnetic fields and the three dimensional usual vector fields of classical electromagnetic theory are given in Grot and Eringen (1966).

We now turn our attention to the source terms $\mathscr{F}^\alpha$, $C^{\alpha\beta}$ and r that appear on the right hand sides of equations (3.15)–(3.21). They represent the long range interactions with exterior fields (eg gravity) and the interactions between matter and electromagnetic fields (eg ponderomotive force and couple). We shall take

$$\left.\begin{aligned} \mathscr{F}^\alpha &= \rho f^\alpha + f^\alpha_{(\mathrm{em})} \\ f^\alpha &= \overline{f^\alpha} + \frac{h}{c^2}u^\alpha \qquad \overline{f^\alpha}u_\alpha = 0 \qquad h = -f^\alpha u_\alpha \end{aligned}\right\} \tag{3.32}$$

$$C^{\alpha\beta} = 2\rho x^{[\alpha}f^{\beta]} + C^{\alpha\beta}_{(\mathrm{em})} \tag{3.33}$$

$$r = \frac{\rho h}{\theta} \tag{3.34}$$

where f^α is the four force per unit mass not caused by the presence of electromagnetic fields, h is the heat supply per unit mass. Among the variety of forms proposed by different authors, the ponderomotive force $f^\alpha_{(\mathrm{em})}$ and the ponderomotive couple $C^{\alpha\beta}_{(\mathrm{em})}$

are chosen to be

$$f^{\alpha}_{(\mathrm{em})} = \frac{1}{c} F^{\alpha}_{.\gamma} J^{\gamma} + \frac{1}{2} \rho \tilde{\Pi}_{\mu\nu} F^{\mu\nu;\alpha} \tag{3.35}$$

$$C^{\alpha\beta}_{(\mathrm{em})} = 2x^{[\alpha} f^{\beta]}_{(\mathrm{em})} + 2\rho F^{[\alpha}_{.\gamma} \tilde{\Pi}^{|\gamma|\beta]}. \tag{3.36}$$

The first term of equation (3.35) is the Lorentz force while the second one is the Stern–Gerlach force in magnetized media. This arbitrary choice, as emphasized by Penfield and Haus (1967), does not bring any damage to the calculated values of observable physical forces since the controversy about the form of $f^{\alpha}_{(\mathrm{em})}$ concerns small relativistic effects. In equations (3.35)–(3.36), we have introduced the magnetization–polarization tensor $\tilde{\Pi}^{\alpha\beta}$ per unit mass by

$$\rho \tilde{\Pi}^{\alpha\beta} \equiv \Pi^{\alpha\beta} = F^{\alpha\beta} - G^{\alpha\beta}. \tag{3.37}$$

$\Pi^{\alpha\beta}$ admits the unique decomposition

$$\left.\begin{aligned} &\Pi^{\alpha\beta} = \frac{1}{c}(\mathscr{P}^{\alpha} u^{\beta} - \mathscr{P}^{\beta} u^{\alpha}) + \frac{1}{\mathrm{i}c} \epsilon^{\alpha\beta\gamma\delta} \mathscr{M}_{\gamma} u_{\delta} \\ &\mathscr{M}_{\gamma} \equiv \frac{1}{2\mathrm{i}c} \epsilon_{\gamma\alpha\beta\delta} \Pi^{\alpha\beta} u^{\delta} \qquad \mathscr{P}^{\alpha} \equiv \frac{1}{c} \Pi^{\beta\alpha} u_{\beta} \\ &\mathscr{M}_{\gamma} u^{\gamma} = 0 \qquad \mathscr{P}^{\alpha} u_{\alpha} = 0. \end{aligned}\right\} \tag{3.38}$$

Then

$$\mathscr{P}^{\alpha} = \mathscr{D}^{\alpha} - \mathscr{E}^{\alpha} \qquad \mathscr{M}^{\alpha} = \mathscr{B}^{\alpha} - \mathscr{H}^{\alpha}.$$

Note that it is easily shown that $f^{\alpha}_{(\mathrm{em})}$ can be written as

$$\left.\begin{aligned} &f^{\alpha}_{(\mathrm{em})} = \frac{1}{c} F^{\alpha}_{.\gamma} J^{\gamma} + \Pi^{\mu\nu} F^{\alpha}_{.\nu;\mu} = -T^{\alpha\beta}_{(\mathrm{em});\beta} \\ &T^{\alpha\beta}_{(\mathrm{em})} \equiv -F^{\alpha}_{.\gamma} G^{\gamma\beta} + \tfrac{1}{4} F_{\mu\nu} F^{\nu\mu} g^{\alpha\beta} \end{aligned}\right\} \tag{3.39}$$

where $T^{\alpha\beta}_{(\mathrm{em})}$ is the electromagnetic energy–momentum tensor.

We have supposed (cf § 1) that the theory presented was not given for polar media. Therefore, the intrinsic spin that appears in equation (3.26, part one) can only be due to magnetization. For an *isotropic* gyromagnetic effect, it is shown that $S^{\alpha\beta}$ and $\tilde{\Pi}^{\alpha\beta}$ are linked by the relation

$$S^{\alpha\beta} = \gamma^{-1} \tilde{\Pi}^{\alpha\beta} \qquad \gamma = \frac{-e}{m_0 c} \tag{3.40}$$

in which γ is the gyromagnetic ratio for an electron (e and m_0 are respectively the charge and the rest mass of the electron). Then equation (3.26, part three) is nothing but the so called Frenkel condition (Frenkel 1926) which asserts that, following the hypothesis of Uhlenbeck and Gouldsmit, the spin is purely magnetic in the rest frame of an electron. On account of equation (3.26, part three), we see that a unique four vector s^{α} referred to as the spin four vector can be associated with $S^{\alpha\beta}$ through the relations

$$s_{\alpha} \equiv \frac{1}{2\mathrm{i}c} \epsilon_{\alpha\beta\gamma\delta} S^{\beta\gamma} u^{\delta} \qquad S^{\beta\gamma} = \frac{1}{\mathrm{i}c} \epsilon^{\beta\gamma\alpha\rho} s_{\alpha} u_{\rho} \qquad s^{\alpha} u_{\alpha} = 0. \tag{3.41}$$

That is, s^{α} is PU; the spin is an axial vector which is space-like. This is in conformity with our ideas on quantum mechanical spin. We remark that equations (3.26, part three) and (3.40) require that $\mathscr{P}^\alpha$ vanishes. The case of dielectric materials is thus eliminated or, the present theory applies to very weakly polarizable media. We shall take in the sequel

$$\mathscr{P}_\alpha \equiv 0 \qquad \text{that is} \qquad \mathscr{E}_\alpha \equiv \mathscr{D}_\alpha. \tag{3.42}$$

It remains to specify the form of the energy ϵ. It is in general a functional of the motion (2.3), of $\tilde{\Pi}^{\alpha\beta}$ (or $\mathscr{M}^\alpha/\rho$) and of θ. Special function approximations of this functional yield constitutive equations for different material classes. A special function form corresponding to elastic materials will be considered in part II. In some cases, it may prove more convenient to use the energy defined by

$$e = \epsilon + \frac{1}{2}\tilde{\Pi}_{\mu\nu}F^{\nu\mu} = \epsilon + \frac{1}{\rho}\mathscr{M}^\alpha\mathscr{B}_\alpha \tag{3.43}$$

in which the term $\frac{1}{2}\tilde{\Pi}_{\mu\nu}F^{\nu\mu}$ is the energy of a magnetic doublet per unit mass.

We are now in a position to write down the local field equations for the theory of deformable magnetized materials endowed with a continuous distribution of electronic spins. In order to obtain these equations in a form closely related to that of three dimensional formalism, every tensorial equation of the set (3.15) through (3.21) is written by using the decomposition of all tensor fields whose four vectorial form is used as much as possible. By this operation, the splitting of space and time that were synthesized in minkowskian formalism is then accomplished. Taking the 'slow motion' limit, that is, the so called nonrelativistic limit, is therefore straightforward.

We shall give only the final results of this operation. In order to help the reader, useful intermediate results are given in Appendix 1.

4. Four vectorial form of the field equations

4.1. Equation of continuity

$$(\rho u^\alpha)_{;\alpha} = 0 \qquad [\rho u^\alpha] n_\alpha = 0. \tag{4.1}$$

4.2. First Cauchy's equations

Upon carrying (3.24) and (3.29) into (3.16, part one), applying the operator $P^\gamma_{.\alpha}$ to the result and taking account of equations (A.1) through (A.6), we obtain three independent equations that correspond to the conservation law of momentum, that is, to the first Cauchy's equations of classical continuum mechanics

$$\begin{aligned}\rho\dot{u}^\gamma = {} & t^{\beta\gamma}{}_{;\beta} + \rho\overline{f^\gamma} + \frac{1}{ic^2}\epsilon^{\gamma\sigma\mu\nu}\mathscr{B}_\mu j_\sigma u_\nu + \frac{1}{2}\Pi_{\mu\nu}F^{\mu\nu;\gamma} \\ & + \frac{1}{c^2}q\mathscr{E}^\alpha + \frac{1}{c^2}\left(\frac{1}{2}\Pi_{\mu\nu}\dot{F}^{\mu\nu}u^\gamma - \rho\epsilon\dot{u}^\gamma - q^\beta u^\gamma{}_{;\beta}\right. \\ & \left. - c^2 p^\gamma u^\beta{}_{;\beta} - c^2\dot{p}^\gamma + p^\alpha\dot{u}_\alpha u^\gamma - u^\gamma t^{\beta\alpha}u_{\alpha;\beta}\right)\end{aligned} \tag{4.2}$$

in which we have gathered within parentheses terms that vanish at the nonrelativistic limit.

4.3. Second Cauchy's equations

Carrying the expressions (3.25), (3.26), (3.33) and (3.36) into equation (3.17, part one) while taking account of equation (3.16, part one), we have

$$\tfrac{1}{2}\rho\dot{S}^{\alpha\beta}+T^{[\beta\alpha]}-M^{\alpha\beta\gamma}{}_{;\gamma} = F^{[\alpha}_{.\gamma}\Pi^{|\gamma|\beta]}. \tag{4.3}$$

Applying the operator $P^{\mu}_{.\beta}P^{\nu}_{.\alpha}$ to this equation, taking account of equations (A.7)–(A.10) and noting that, from equation (3.24)

$$T^{[\beta\alpha]} = \frac{1}{c^2}u^{[\beta}q^{\alpha]}+p^{[\beta}u^{\alpha]}-t^{[\alpha\beta]} \tag{4.4}$$

we obtain the projection of equation (4.3) onto $M^3_\perp$

$$\frac{\rho}{2}\dot{S}^{\nu\mu} = t^{[\nu\mu]}+M^{\nu\mu\gamma}{}_{;\gamma}+\mathscr{M}^{[\nu}\mathscr{B}^{\mu]}+\frac{1}{c^2}(\rho S^{[\nu}_{.\alpha}u^{\mu]}\dot{u}^{\alpha}-2M^{[\nu|\beta\gamma|}u^{\mu]}u_{\beta;\gamma}). \tag{4.5}$$

The term within parentheses disappears at the nonrelativistic limit. Note that only three of equations (4.5) are independent. They correspond to the conservation of moment of momentum, that is, to the second Cauchy's equations of classical continuum mechanics. Alternatively, equations (4.5) define $t^{[\nu\mu]}$. If there are neither magnetic spins nor couple stresses, these equations reduce to those of Grot and Eringen (1966, equation (5.19) with $\mathscr{P}^{\alpha} = 0$)†.

It is also of interest to project equation (4.3) along u^{α}. On contracting equation (4.3) with u_{α} and using (4.4), (2.4), (2.5), (3.24, parts two to four) (3.30) and (3.38), one gets an expression for the nonmechanical momentum p^{β}

$$p^{\beta} = \frac{1}{c^2}q^{\beta}+\frac{\rho}{c}S^{\beta\alpha}\dot{u}_{\alpha}+\frac{2}{c}M^{\alpha\beta\gamma}u_{\alpha;\gamma}-\frac{1}{\mathrm{i}c^2}\epsilon^{\beta\gamma\mu\nu}u_{\nu}\mathscr{M}_{\mu}\mathscr{E}_{\gamma} \tag{4.6}$$

where one used the identities

$$\dot{S}^{\alpha\beta}u_{\beta} \equiv -S^{\alpha\beta}\dot{u}_{\beta} \qquad M^{\alpha\beta\gamma}{}_{;\gamma}u_{\alpha} \equiv -M^{\alpha\beta\gamma}u_{\alpha;\gamma} \tag{4.7}$$

which follow from the PU character of $S^{\alpha\beta}$ and $M^{\alpha\beta\gamma}$. Only three of equations (4.6) are independent. In absence of heat, we see that p^{β} is due only to electromagnetic fields and to the spin.

4.4. Energy equation

This equation results from the projection of equation (3.16, part one) along u^{α}. With p^{α} and $t^{\beta\alpha}$ PU and using equations (A.11)–(A.12) and (2.4, part three) and (2.5, part two), we obtain the scalar equation

$$\rho\dot{\epsilon}+q^{\beta}{}_{;\beta}+p^{\alpha}\dot{u}_{\alpha}-t^{\beta\alpha}u_{\alpha;\beta} = \rho h+\mathscr{E}_{\gamma}j^{\gamma}-\mathscr{M}^{\alpha}\dot{\mathscr{B}}_{\alpha}-\frac{1}{c}\mathscr{E}_{\alpha}\overset{*}{\Pi}{}^{\beta\alpha}\dot{u}_{\beta}. \tag{4.8}$$

Further transformations of this equation will be given later on.

† Within an alternation of the indices of $t^{\alpha\beta}$. This comes from a different decomposition of $T^{\alpha\beta}$.

4.5. Entropy inequality

Substituting the expressions (3.27) and (3.34) into equation (3.18, part one), we get

$$\rho\dot{\eta}+\frac{1}{\theta}q^{\alpha}{}_{;\alpha}-\frac{1}{\theta^{2}}q^{\alpha}\theta_{,\alpha}+\frac{\rho h}{\theta}\geqslant 0. \tag{4.9}$$

4.6. Electromagnetic equations

By projection along u^{α} and onto $M_{\perp}^{3}$ of equations (3.19) through (3.21), bearing in mind equation (3.42), one gets in $(\mathscr{B}-\Sigma)$

$$j^{\alpha}{}_{;\alpha}+\dot{j}+\frac{q}{c^{2}}u^{\alpha}{}_{;\alpha}=0 \tag{4.10}$$

$$P^{\gamma}_{.\beta}\mathscr{B}^{\beta}{}_{;\gamma}-\frac{1}{ic}\epsilon^{\alpha\beta\gamma\mu}\mathscr{E}_{\alpha}u_{\beta;\gamma}u_{\mu}=0 \tag{4.11}$$

$$\frac{1}{ic}\epsilon^{\alpha\beta\gamma\mu}\mathscr{E}_{\gamma;\beta}u_{\mu}-\frac{1}{ic}\epsilon^{\alpha\beta\gamma\mu}\mathscr{E}_{\beta}\dot{u}_{\gamma}u_{\mu}+P^{\alpha}_{.\beta}\dot{\mathscr{B}}^{\beta}+\mathscr{B}^{\alpha}u^{\beta}{}_{;\beta}-\mathscr{B}^{\beta}u^{\alpha}{}_{;\beta}=0 \tag{4.12}$$

$$P^{\gamma}_{.\beta}\mathscr{E}^{\beta}{}_{;\gamma}+\frac{1}{ic}\epsilon^{\alpha\beta\gamma\mu}\mathscr{H}_{\alpha}u_{\beta;\gamma}u_{\mu}=q \tag{4.13}$$

$$\frac{1}{ic}\epsilon^{\alpha\beta\gamma\mu}\mathscr{H}_{\gamma;\beta}u_{\mu}-\frac{1}{ic}\epsilon^{\alpha\beta\gamma\mu}\mathscr{H}_{\beta}\dot{u}_{\gamma}u_{\mu}-P^{\alpha}_{.\beta}\dot{\mathscr{E}}^{\beta}+\mathscr{E}^{\beta}u^{\alpha}{}_{;\beta}-\mathscr{E}^{\alpha}u^{\beta}{}_{;\beta}=j^{\alpha} \tag{4.14}$$

respectively. Only three of equations (4.12) are independent. The same is true of equations (4.14). The above equations are the obvious four vectorial of Maxwell's equations, more specifically of the charge conservation, the equation of nonexistence of magnetic poles, the equations of Faraday, Ampère and Gauss respectively.

5. Saturated magnetization

So far no hypotheses have been made as to the magnitude of the magnetization. Of importance for the study of ferromagnetic materials, is the case for which the magnetization three vector has a constant magnitude over the specimen of material considered. This situation happens in ferromagnetic media brought below the Curie temperature. Then some materials present a structure in domains in each of which the magnetization vector is constant in magnitude and in direction. An overall look at the specimen shows that the magnitude of the magnetization is constant throughout the whole specimen but its direction varies from one domain to another, thus yielding a total zero (more exactly, a negligibly small total) spontaneous magnetization in the multidomain crystal or polycrystalline material in this equilibrium state. Clearly this state can exist without any applied magnetic field. The application of such a magnetic field perturbs the distribution of magnetic moments. A rotation of the magnetization vector is observed within each domain while the latter is deformed to yield a new state of equilibrium. During the rotation, the magnetic moment has to overcome different actions. First, the magnetization has a preference for a certain orientation with respect to crystalline axes. A certain amount of energy referred to as the *anisotropy energy* is thus used. Second, the motion of each magnetic moment is linked to that of its neighbours. This effect gives rise to the notion of *exchange forces*. Finally, due to the dissipation of which the nature is not clearly known, the rotation of the magnetic moments suffers a damping (cf Brown 1966, Maugin and Eringen 1972a).

A deformable magnetically saturated material may be represented by an ensemble of material points to each of which is attached a magnetic moment μ^k per unit mass, of constant magnitude throughout the body (B) $[(B_R)$ in the reference space $E_R^3]$. We have

$$\mu^k \mu_k = \mu_S^2 = \text{constant}. \tag{5.1}$$

This equation may be given a covariant four dimensional form. We remark that, from the decomposition of the tensor $\tilde{\Pi}^{\alpha\beta}$, we have in a rest frame

$$\tilde{\Pi}^{\alpha\beta} \stackrel{*}{=} [\text{dual}(\mu^k), ip^k] \tag{5.2}$$

where p^k is the polarization three vector in E^3. From equations (3.40) and (3.26, part three), it follows that:

$$S^{\alpha\beta} \stackrel{*}{=} [\gamma^{-1}\, \text{dual}(\mu^k), 0]. \tag{5.3}$$

Thus, the covariant form of equation (5.1) reads

$$\tfrac{1}{2} S_{\alpha\beta} S^{\beta\alpha} = S_0^2 = \text{constant} = s_\alpha s^\alpha = s_0^2 \tag{5.4}$$

in which we have made use of equations (3.41). Taking the proper time rate of equation (5.4), we get

$$\dot{S}_{\alpha\beta} S^{\beta\alpha} = \dot{s}_\alpha s^\alpha = 0. \tag{5.5}$$

Then, contracting equation (4.6) with $S_{\beta\alpha}$ and taking account of equations (3.26, part three) and (4.4), we obtain the following scalar equation which can be considered as a constraint during the motion of a deformable magnetically saturated medium:

$$S_{\beta\alpha}(t^{\alpha\beta} + M^{\alpha\beta\gamma}{}_{;\gamma}) = 0 \tag{5.6}$$

since

$$S_{\beta\alpha} F^{[\alpha}_{\cdot\gamma} \Pi^{|\gamma|\beta]} = 0 \tag{5.7}$$

from the skewsymmetry of $F^{\alpha\gamma}$.

From the condition (5.4), *a priori* forms of the equation (4.3) describing the motion of the magnetization can be given. Let $\Omega^{\alpha\beta}$ be the skewsymmetric tensor representing the rotation of s^α in M^4. Then such a form is

$$\dot{S}^{\alpha\beta} = 2\Omega^{[\alpha}_{\cdot\gamma} S^{|\gamma|\beta]} - \lambda\{\tfrac{1}{2}\Omega^{\alpha\beta} - \tfrac{1}{2} S_0^{-2}(\tfrac{1}{2}\Omega_{\mu\nu} S^{\nu\mu}) S^{\alpha\beta}\} \tag{5.8}$$

where λ is a constant coefficient. On contraction of this equation with $S_{\beta\alpha}$, we get equation (5.5). On contraction with $\Omega_{\beta\alpha}$, we get

$$\dot{S}^{\alpha\beta} \Omega_{\beta\alpha} \equiv 0. \tag{5.9}$$

This equation shows that in a real displacement $\Omega_{\beta\alpha}$, the magnetic spin does not work. The quantity $\dot{S}^{\alpha\beta}$ is a D'Alembertian–inertia couple (cf Tiersten 1964) or, in other words, the magnetic spin presents features of gyroscopical nature.

Another possible form of equation is the following:

$$\dot{S}^{\alpha\beta} = 2\Omega^{[\alpha}_{\cdot\gamma} S^{|\gamma|\beta]} + \frac{\alpha}{S_0} S^{[\alpha}_{\cdot\gamma} \dot{S}^{|\gamma|\beta]} \tag{5.10}$$

where α is a constant coefficient. Here also, it can be shown that equations (5.5) and (5.9) follow. Equations (5.8) and (5.10) represent the four dimensional relativistic equivalent forms of the equations given by Landau and Lifshitz (1935) and Gilbert and Kelley (1955) respectively. In both equations, the second term of the right hand side describes the approach of the magnetization into parallelism with the *effective magnetic field* resulting from the applied magnetic field and the different interactions. These terms introduce a dissipation and yield a noncoplanar rotation of the magnetization. They are not equivalent except in certain approximations (cf Maugin and Eringen, 1972a). We shall see in part II how equation (4.3) and equation (5.8) or (5.10) can be shown to be equivalent on a correct definition of the effective field mentioned above.

6. Conclusion

The equations (4.1), (4.2), (4.5), (4.8), (4.10)–(4.14) constitute the set of field equations governing the behaviour of a deformable *magnetized* and *weakly polarizable* material endowed with electronic spins in $(\mathscr{B})$. The number of independent components amounts to: $1+3+6+1+1+8 = 20$. They are supplemented with corresponding jump relations across the discontinuity hypersurface (Σ) and boundary conditions on $(\partial\mathscr{B})$. The unknown fields of the problem are u^α, ρ, $t^{\beta\alpha}$, q^β, ϵ (or e), j^α, $\mathscr{B}^\alpha$, $\mathscr{H}^\alpha$ (or $\mathscr{M}^\alpha$), $\mathscr{E}^\alpha$ and $M^{\alpha\beta\gamma}$ of which the number of independent components is respectively 3, 1, 16, 3, 1, 3, 3, 3, 3 and 24 thus totaling 60 unknown scalar quantities. We see that the system is underdetermined. We need 46 equations more to close the system. This is precisely the total number of independent components of constitutive equations for the fields j^α, $t^{\beta\alpha}$, q^β and $M^{\alpha\beta\gamma}$. Part II of this work will be devoted to the study of such constitutive equations for a specified class of materials with the help of the constraint provided by the inequality (4.9) of which we shall give several forms.

Appendix 1

The following expressions may be readily verified:

$$\dot{u}^\alpha P^\gamma_{.\alpha} \equiv \dot{u}^\gamma \qquad u^\alpha_{;\beta} P^\gamma_{.\alpha} \equiv u^\gamma_{;\beta} \tag{A.1}$$

$$p^\alpha_{;\beta} u^\beta P^\gamma_{.\alpha} = \dot{p}^\gamma - \frac{1}{c^2} p^\alpha \dot{u}_\alpha u^\gamma \tag{A.2}$$

$$P^\gamma_{.\alpha} t^{\beta\alpha}_{;\beta} = t^{\beta\gamma}_{;\beta} - \frac{1}{c^2} u^\gamma t^{\beta\alpha} u_{\alpha;\beta} \tag{A.3}$$

$$\frac{1}{c} F^\alpha_{.\sigma} J^\sigma = \frac{1}{c^2} \mathscr{E}_\sigma j^\sigma u^\alpha + \frac{1}{ic^2} \epsilon^{\alpha\sigma\mu\nu} \mathscr{B}_\mu j_\sigma u_\nu + \frac{1}{c^2} q \mathscr{E}^\alpha \tag{A.4}$$

$$\frac{1}{c} F^\alpha_{.\sigma} J^\sigma P^\gamma_{.\alpha} = \frac{1}{c^2} \epsilon^{\gamma\sigma\mu\nu} \mathscr{B}_\mu j_\sigma u_\nu + \frac{1}{c^2} q \mathscr{E}^\alpha \tag{A.5}$$

$$\frac{1}{2} \Pi_{\mu\nu} F^{\mu\nu;\alpha} P^\gamma_{.\alpha} = \frac{1}{2} \Pi_{\mu\nu} F^{\mu\nu;\gamma} + \frac{1}{2c^2} \Pi_{\mu\nu} \dot{F}^{\mu\nu} u^\gamma \tag{A.6}$$

$$F_{\mu\beta}\Pi^{\alpha\beta} = \frac{1}{c}\overset{*}{\Pi}{}^{\alpha\beta}\mathscr{E}_\beta u_\mu - \frac{1}{c^2}\mathscr{P}^\beta\mathscr{E}_\beta u^\alpha u_\mu + \mathscr{P}^\alpha\mathscr{E}_\mu$$

$$-\frac{1}{c}\overset{*}{F}_{\mu\beta}\mathscr{P}^\beta u^\alpha + P^\alpha_{.\mu}\mathscr{B}^\beta\mathscr{M}_\beta - \mathscr{B}^\alpha\mathscr{M}_\mu \tag{A.7}$$

$$F^{[\alpha}_{.\gamma}\Pi^{|\gamma|\beta]} = -\frac{1}{c}u^{[\alpha}\overset{*}{\Pi}{}^{\beta]\gamma}\mathscr{E}_\gamma - \mathscr{E}^{[\alpha}\mathscr{P}^{\beta]} + \mathscr{M}^{[\alpha}\mathscr{B}^{\beta]} + \frac{1}{c}\overset{*}{F}{}^{[\alpha|\gamma|}\mathscr{P}_\gamma u^{\beta]} \tag{A.8}$$

$$P^\mu_{.\beta}P^\nu_{.\alpha}\dot{S}^{\alpha\beta} = \dot{S}^{\nu\mu} + \frac{2}{c^2}S^{[\mu}_{.\alpha}u^{\nu]}\dot{u}^\alpha \tag{A.9}$$

$$P^\mu_{.\beta}P^\nu_{.\alpha}M^{\alpha\beta\gamma}{}_{;\gamma} = M^{\nu\mu\gamma}{}_{;\gamma} - \frac{2}{c^2}M^{[\nu|\beta\gamma|}u^{\mu]}u_{\beta;\gamma} \tag{A.10}$$

$$\frac{1}{c}F^\alpha_{.\gamma}J^\gamma u_\alpha \equiv -\mathscr{E}_\gamma j^\gamma \tag{A.11}$$

$$\Pi^{\mu\nu}F^\alpha_{.\nu;\mu}u_\alpha \equiv \mathscr{P}^\alpha\dot{\mathscr{E}}_\alpha + \mathscr{M}^\alpha\dot{\mathscr{B}}_\alpha + \frac{1}{c}\mathscr{P}^\alpha\overset{*}{F}_{\beta\alpha}\dot{u}^\beta + \frac{1}{c}\mathscr{E}_\alpha\overset{*}{\Pi}{}^{\beta\alpha}\dot{u}_\beta. \tag{A.12}$$

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